LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS

EXAM 3 - MTH 207: DISCRETE STRUCTURES 1 – FALL 2011

DURATION: 75 MIN

NAME: KEY ID:

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INSTRUCTIONS: This exam consists of 8 pages and 7 problems. Check that none is missing. Answer the questions in the space provided for each problem; if more space is needed, you may use the back pages.

GOOD LUCK!

QUESTION	GRADE
1. 8%	
2. 15%	
3. 15%	
4. 15%	
5. 12%	
6. 26%	
7. 9%	
TOTAL	

1. **(8%)** Find two **nonzero** 3×3 matrices A and B such that $AB = 0$ (the 3×3 zero matrix).

There are many examples –

ex: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

a. **(6%)** Find $A \vee A$ and $A \wedge A$.

$A \vee A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A$

$A \wedge A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A$.

b. **(9%)** Find A^2, A^3, A^4 and guess a formula for A^n

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $A^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $A^4 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Guess: $A^n = \begin{bmatrix} 1 & 1 & (2n-1) \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

c. Bonus: Prove your guess in part b using mathematical induction

(Do this question on the back of the page)

3. Let A denote any square matrix.

a. **(9%)** Define $B = \frac{1}{2}\left(A + A^T\right)$ and $C = \frac{1}{2}\left(A - A^T\right)$. Show that $A = B + C$, that $B^T = B$ (i.e. B is a symmetric matrix) and $C^T = -C$ (i.e. C is an anti-symmetric matrix).

$$B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
$$= \frac{1}{2}\left[2A\right] = A$$

$$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left(A^T + (A^T)^T\right) = \frac{1}{2}(A^T + A) = B$$

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}\left(A^T - A\right) = -C.$$

b. **(6%)** Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. Write $A = B + C$ where B is symmetric and C is anti-symmetric.

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}\right)$$

$$\Rightarrow B = \frac{1}{2}\begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3/2 \\ 1 & 1 & 2 \\ 3/2 & 2 & 1 \end{bmatrix}.$$

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3/2 \\ 1 & 0 & -2 \\ -\frac{3}{2} & 2 & 0 \end{bmatrix}$$

4. Let $A = Z$ and let R denote the relation on A defined by: aRb if $a^2 \equiv b^2 \pmod{6}$

a. **(9%)** Show that this relation is an equivalence relation.

Reflexive: aRa since $a^2 - a^2 = 0 = (0).6$

Symmetric: Suppose $aRb \Rightarrow a^2 \equiv b^2 \pmod{6}$

$\Rightarrow a^2 - b^2 = 6k$

$\Rightarrow b^2 - a^2 = -6k = 6(k')$

$\Rightarrow b^2 \equiv a^2 \pmod{6}$.

Transitive: Suppose aRb and bRc

$$\Rightarrow \left.\begin{array}{l} a^2 \equiv b^2 \pmod{6} \\ b^2 \equiv c^2 \pmod{6} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a^2 - b^2 = 6k_1 \\ b^2 - c^2 = 6k_2 \end{array}\right.$$

Add: $a^2 - c^2 = 6(k_1 + k_2)$

$\therefore a^2 \equiv c^2 \pmod{6}$.

b. **(6%)** Find all its equivalence classes.

Mod 6, $\mathbb{Z}$ is divided into 6 groups:

$6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$,

and $6k+5$.

Square each:

$$(6k)^2 = 36k^2 \equiv 0 \pmod 6$$

$$(6k+1)^2 = 36k^2 + 12k + 1 \equiv 1 = 1^2 \pmod 6$$

$$(6k+2)^2 = 36k^2 + 24k + 4 \equiv 4 = 2^2 \pmod 6.$$

$$(6k+3)^2 = 36k^2 + 36k + 9 \equiv 3^2 \pmod 6.$$

$$(6k+4)^2 = 36k^2 + 48k + 16 \equiv 4 = 2^2 \pmod 6$$

$$(6k+5)^2 = 36k^2 + 60k + 25 \equiv 1 = 1^2 \pmod 6$$

$\Rightarrow$ $[0] = \{6k;\ k \in \mathbb{Z}\}$

$[1] = \{6k+1;\ 6k+5;\ k \in \mathbb{Z}\}$

$[2] = \{6k+2,\ 6k+4;\ k \in \mathbb{Z}\}$

$[3] = \{6k+3;\ k \in \mathbb{Z}\}$.

5. Let R and S denote two relations on a set A.

a. **(6%)** Show that if R and S are transitive then $R \cap S$ is also transitive.

Suppose (x,y) and $(y,z) \in R \cap S$

(x,y) and $(y,z) \in R \Rightarrow (x,z) \in R$ since it is transitive

(x,y) and $(y,z) \in S \Rightarrow (x,z) \in S$ since it is transitive

$\therefore (x,z) \in R \cap S$

$\therefore R \cap S$ is transitive.

b. **(6%) It is a fact that if** R and S are transitive **then** $R \cup S$ is **not** transitive: Prove this statement either by giving an example or else run through a proof until it fails.

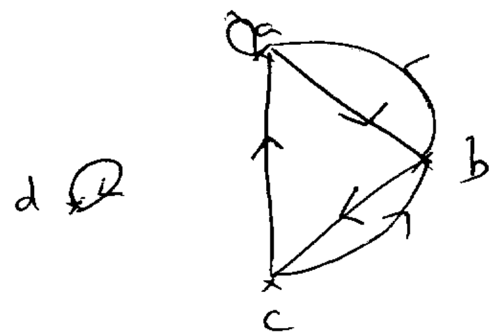
Ex: $R = \{(1,1);\ (1,2);\ (2,1)\}$ is transitive

$S = \{(2,2);\ (2,3);\ (3,2)\}$ is transitive

But $R \cup S$ is not transitive.

6. Let $A = \{a, b, c, d\}$ and $M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a. **(6%)** Draw the digraph associated with this matrix.



b. **(6%)** Find all the paths of length 2 from vertex a to each of the other vertices.

$a \to a \to b$

$a \to b \to c$

$a \to a \to a$ <u>or</u> $a \to b \to a$

No path of length 2 from a to d.

c. **(8%)** Evaluate M^2. What do the entries of M^2 represent?

$$M^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$; they represent the number of paths from length 2 from one vertex to another

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d. **(6%)** Find the relation R corresponding to M (write this relation as a set of pairs)

$$R = \{(a,a); (a,b); (b,a); (b,c); (c,a); (c,b); (d,d)\}$$

7. **(9%)** Let $S = \{2,4,6,8,\ldots\}$ be the set of positive integers. Define the relation R on S by: mRn if $\frac{m}{n}$ is an even integer. Check if this relation is reflexive, symmetric, or transitive.

Reflexive: $\frac{m}{m} = 1$ not even $\Rightarrow$ not reflexive

Symmetric: if $\frac{m}{n} = k \Rightarrow \frac{n}{m} = \frac{1}{k}$ not an integer

$\therefore$ not symmetric.

Transitive: if $\frac{m}{n} = k_1$ and $\frac{n}{p} = k_2$,

then $\frac{m}{p} = \frac{m}{n} \cdot \frac{n}{p} = k_1 k_2 \in \mathbb{Z}$

$\therefore$ it is transitive.